Cost minimization with a quasilinear production function

Consider the following production function:

$$x = L + 10\sqrt{K}$$

where L and K denote the quantities of labor and capital, respectively. Derive the firm's cost function.

Solution

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Let w be the wage rate and r the rental rate of capital. To produce a given output level x, the firm solves the cost minimization problem

$$\min_{L,K} \; wL + rK$$

subject to

$$x = L + 10\sqrt{K}$$

We are asked to derive only the interior solution, so we assume

$$L > 0 \qquad K > 0$$

The Lagrangian is

$$\mathcal{L} = wL + rK + \lambda\left(x - L - 10\sqrt{K}\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \frac{5}{\sqrt{K}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x - L - 10\sqrt{K} = 0$$

From the first condition,

$$\lambda = w$$

Substituting into the second condition,

$$r = w\frac{5}{\sqrt{K}}$$

$$\sqrt{K} = \frac{5w}{r}$$

$$K^* = \frac{25w^2}{r^2}$$

Using the production constraint,

$$L^* = x - 10\sqrt{K^*}$$

$$L^* = x - 10\left(\frac{5w}{r}\right)$$

$$L^* = x - \frac{50w}{r}$$

For the interior solution to be valid, we need

$$L^* > 0$$

that is,

$$x > \frac{50w}{r}$$

Now substitute the optimal inputs into the cost function

$$C(w,r,x) = wL^* + rK^*$$

$$C(w,r,x) = w\left(x - \frac{50w}{r}\right) + r\left(\frac{25w^2}{r^2}\right)$$

$$C(w,r,x) = wx - \frac{50w^2}{r} + \frac{25w^2}{r}$$

$$C(w,r,x) = wx - \frac{25w^2}{r}$$

Thus, the interior conditional factor demands are

$$L^*(w,r,x) = x - \frac{50w}{r}$$

$$K^*(w,r,x) = \frac{25w^2}{r^2}$$

and the corresponding interior cost function is

$$C(w,r,x) = wx - \frac{25w^2}{r}$$

valid for

$$x > \frac{50w}{r}$$

Corner solution

The interior solution is valid only when

$$L^* = x - \frac{50w}{r} \geq 0$$

that is,

$$x \geq \frac{50w}{r}$$

If instead

$$x < \frac{50w}{r}$$

the interior solution is not feasible, so the optimum is at the boundary

$$L = 0$$

Using the production constraint,

$$x = 10\sqrt{K}$$

we get

$$K^* = \frac{x^2}{100}$$

Hence, the corner conditional factor demands are

$$L^*(w,r,x) = 0$$

$$K^*(w,r,x) = \frac{x^2}{100}$$

and the corresponding corner cost function is

$$C(w,r,x) = \frac{rx^2}{100}$$

Therefore, the full cost function is

$$C(w,r,x) = \begin{cases} \frac{rx^2}{100} & \text{if } x \leq \frac{50w}{r} \\ wx - \frac{25w^2}{r} & \text{if } x \geq \frac{50w}{r} \end{cases}$$

The two branches coincide at

$$x = \frac{50w}{r}$$

so the cost function is continuous